Lorentzian Manifolds and Causal Sets as Partially Ordered Measure Spaces

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Abstract

We consider Lorentzian manifolds as examples of partially ordered measure spaces, sets endowed with compatible partial order relations and measures, in this case given by the causal structure and the volume element defined by each Lorentzian metric. This places the structure normally used to describe spacetime in geometrical theories of gravity in a more general context, which includes the locally finite partially ordered sets of the causal set approach to quantum gravity. We then introduce a function characterizing the closeness between any two partially ordered measure spaces and show that, when restricted to compact spaces satisfying a simple separability condition, it is a distance. In particular, this provides a quantitative, covariant way of describing how close two manifolds with Lorentzian metrics are, or how manifoldlike a causal set is.

1. Introduction

On a spacetime manifold M, assigning a Lorentzian metric tensor field is equivalent to specifying at each $x \in M$ the set of null directions (the "shape of the light cone") and the determinant of the metric (the volume element); this is a local statement that can be easily checked (one just needs to be careful and take into account the Lorentz symmetry group of the metric in this correspondence). There is also a global version of this statement, a result whose proof [1, 2] is not as simple, which states that if the spacetime is *past and future distinguishing*,[1] then the causal relations among its points determine uniquely the spacetime topology, differentiable structure and conformal metric;[2] adding the volume element would then uniquely specify the metric. In other words, the geometry of a classical distinguishing spacetime can be fully captured by describing it as a set with a partial order relation among its points encoding all of the causal information, and a measure with which volumes can be determined. This is the structure we will work with; for convenience, we will add a simple compatibility condition between the partial order and the measure that is always satisfied when the set in question is a Lorentzian manifold.

We take a partially ordered set to be a pair $(X, \prec)$ of a set X and a binary relation $\prec$ on X which is transitive (if $x \prec y$ and $y \prec z$ then $x \prec z$) and irreflexive (for all $x \in X$, $x \not\prec x$). This is often referred to as an *irreflexive* (or *strict*) partial order, as opposed to a *reflexive* one which satisfies $x \prec x$ for all x and antisymmetry (if $x \prec y$ and $y \prec x$ then $x = y$), and is more convenient to use together with the Lorentzian distance we define below. A measure on a set X is a non-negative, countably additive function $\mu : \mathcal{A} \to \mathbb{R}^+$ on a σ-algebra of subsets of X,[3] that are therefore called measurable subsets. We call *partially ordered measure space* a quadruple $(X, \prec, \mathcal{A}, \mu)$ of a set X with a partial order relation $\prec$ and a measure μ on a σ-algebra $\mathcal{A}$ of subsets of X, such that for all x, $y \in X$ the pasts and futures

$$I^-(x) := \{z \in X \mid z \prec x\}\ , \qquad I^+(x) := \{z \in X \mid x \prec z\} \tag{1}$$

are measurable, i.e., they belong to $\mathcal{A}$. It then follows that the Alexandrov sets (or intervals)

$$A(x,y) := \{z \in X \mid x \prec z \prec y\} \equiv \big(I^+(p) \cap I^-(q)\big) \tag{2}$$

are also measurable. Since our interpretation of the partial order on X is related to causality, as well as for brevity, we will also refer to $(X, \prec, \mathcal{A}, \mu)$ as a *causal measure space*.

Every chronological Lorentzian manifold (M, g) (i.e., one in which there are no closed timelike curves) is a causal measure space. In this case, the partial order we use is defined by the statement that $x \prec y$ iff y is in the chronological future of x (which is why we chose the notation $I^\pm(x)$ in (1)); $\mathcal{A}$ is the usual Borel σ-field of the manifold M, generated by

[1] This is a relatively mild causality condition that can be interpreted as stating that there are no closed or almost closed timelike curves (see, e.g., Ref [3]); in a Lorentzian manifold, it is equivalent to the requirement that no two distinct points in M have the same chronological past or future.

[2] It is interesting to note that in the case of distinguishing spacetimes the causal relations are in turn determined by the *horismos* relations [4], so in fact the topological, differentiable and conformal structures can be obtained from the latter relations, more closely analogous to the null cone of the local statement.

[3] Thus, $\mathcal{A}$ is a non-empty collection of subsets which is closed under complementation (if $A \in \mathcal{A}$ then $X \setminus A \in \mathcal{A}$) and countable unions (if $A_i \in \mathcal{A}$, $i \in \mathbb{N}$, then $\bigcup_{i=1}^\infty A_i \in \mathcal{A}$), and μ is countably additive if for disjoint A_i, $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$.

the open sets in the manifold topology; $\mu(R)$ is the volume of a measurable region $R \subset M$, $\mu(R) := V(R) = \int_R \sqrt{-g}\, \mathrm{d}^d x$ in a d-dimensional manifold; and the compatibility condition that pasts and futures be measurable is satisfied simply by virtue of the fact that they are always open subsets of M. If no additional causality condition were imposed, it would be possible for two different Lorentzian manifolds to give the same causal measure space. In this paper, however, we will usually restrict our attention to continuum spacetimes that are distinguishing, and it follows from results quoted above that two distinguishing Lorentzian manifolds are isomorphic iff they are isomorphic as causal measure spaces. In terms of the measure, in Lorentzian geometry one normally imposes that metrics be non-degenerate; in our case, however, we can allow the determinant g to vanish at isolated points or subsets of M of codimension at least 1.

Causal measure spaces can be much more general than differentiable manifolds with Lorentzian metrics. For example, they can be discrete as in the causal set approach to quantum gravity [5].[4] In the language of this paper, a causal set can be defined as a causal measure space on a set X that is at most countable, with $\mathcal{A}$ the power set 2^X, μ the "counting" measure which gives the cardinality of a subset,

$$\forall B \subseteq X \quad \mu(B) = |B| \,, \tag{3}$$

and X locally finite in the sense that all Alexandrov sets have finite measure, $|A(x,y)| < \infty$. The distinguishing condition, in the form

$$x \neq y \;\Rightarrow\; I^{\pm}(x) \neq I^{\pm}(y) \,, \tag{4}$$

could be meaningfully imposed on any causal measure space, but we will not do so in the case of causal sets, because that would exclude too many of them (for example some of the simplest causal sets, such as the 3-element V-shaped one, its Λ-shaped time-reversal, or the 4-element diamond-shaped one, where the two unrelated elements have the same pasts and futures), and because in the discrete case a set that does not satisfy (4) does not necessarily have almost closed timelike curves. About the measure, in the discrete case we will assume that all elements of the set have nonzero measure, $\mu(\{x\}) > 0$.

Causal sets are used to model quantum spacetime at small scales, and the fact that they can be included in the same class of structures as Lorentzian manifolds, which model classical spacetime at macroscopic scales, is a major part of our motivation for studying causal measure spaces. However, in order to conclude that this structure really is useful in the causal-set approach to quantum gravity, we need to show that it provides the right type of tools for the kinematics of the theory. In particular we need to show that, despite the fact that causal sets and Lorentzian manifolds are described by very different formalisms, there are causal sets that at large scales are basically indistinguishable, in a quantitative sense, from a classical spacetime [9]. What we intend to do in the rest of this paper is to start addressing this point, by introducing structure on the set of all causal measure spaces that will allow us to quantify the extent to which any two of them, in particular causal sets and Lorentzian manifolds, resemble each other.

The rest of the paper is organized as follows. In Sec 2 we introduce a Lorentzian distance on an arbitrary causal measure space and use it in Sec 3 to define a closeness function between

[4] For conceptually oriented reviews, see Refs [6, 7, 8]; for overviews of some results see also Refs [9, 10, 11].

causal measure spaces. Earlier proposals for a closeness measure between Lorentzian metrics on a fixed manifold [12], between Lorentzian geometries [13], and an actual distance between Lorentzian geometries [14] have appeared in the literature. However, even in the case of the functions that compared Lorentzian geometries, one was restricted to continuum manifolds [14] and the other one was defined in probabilistic terms that made it difficult to work with [13]. The proposal in this paper overcomes the first limitation by construction, because it is directly formulated in terms of a more general class of objects that include causal sets, and it allows us to more easily prove our main result because it is not probabilistic. In Sec 4 we introduce an auxiliary pseudodistance on causal measure spaces, and consider some useful examples for the purpose of understanding properties of our closeness function. In Sec 5 we show our main result, that when restricted to a suitably defined class of causal measure spaces our closeness function is in fact a distance. Sec 6 contains some concluding remarks, focusing on the relevance of our framework for causal set theory as an approach to quantum gravity, and in particular how our distance can help us define manifoldlike causal sets and make progress in the main open issues in causal set kinematics.

2. Causal measure spaces and Lorentzian distances

We will call Lorentzian distance on a set X a two-point function $d : X \times X \to \mathbb{R}^+$ which:
(i) Is "one-sided" in the sense that, $\forall x, y$, if $d(x,y) > 0$ then $d(y,x) = 0$; In particular, for consistency we need to have that $d(x,x) = 0$ for all x; and
(ii) Satisfies the "reverse triangle inequality", the statement that

$$\forall x, y, z, \quad \text{if} \quad d(x,y)\, d(y,z) > 0 \quad \text{then} \quad d(x,y) + d(y,z) \leq d(x,z)\ .$$

There is a natural notion of compatibility between Lorentzian distances and partial order relations: we can say that d and $\prec$ are weakly compatible if whenever $d(x,y) > 0$ we have $x \prec y$, and strongly compatible if the implication holds both ways, $x \prec y$ iff $d(x,y) > 0$. In fact, given a Lorentzian distance d on X we can use the latter condition to define a partial order on X. On the other hand, given just a partial order on a set X there is no general way of defining a natural Lorentzian distance on X, although in the purely discrete case one could use for $d(x,y)$ the length of the longest maximal chain between x and y, and in the case of a Lorentzian manifold the supremum of the length of all future-directed causal curves between x and y (or zero if such curves don't exist) has been used in the literature [15, 14, 16] as a Lorentzian distance.[5]

If the set X is endowed with a causal measure space structure, however, the correspondence is a closer one. A natural Lorentzian distance between elements x and y of a causal measure space is the measure of the interval between them. Specifically, we define

$$d(x,y) := \begin{cases} \mu(A(x,y) \cup \{y\}) & \text{if } x \prec y\ , \\ 0 & \text{otherwise}\ . \end{cases} \tag{5}$$

[5] Notice that there is a mismatch between those two definitions of Lorentzian distance, in the sense that the length of the longest chain does not correspond directly to the length of the longest geodesic in the continuum limit for manifoldlike causal sets, although the two are related [17]. This, together with the fact that the extension would be ambiguous for spaces that are neither causal sets nor Lorentzian manifolds, are the reasons why we do not generalize the longest chain/geodesic definition to general causal measure spaces.

This d and the partial order are obviously weakly compatible, since $d(x, y) = 0$ unless $x \prec y$. The reason for adding the single-element set $\{y\}$ to $A(x, y)$ is to ensure that, when restricted to causal sets, $d(x, y) > 0$ whenever $x \prec y$, so that d can maximally distinguish elements and is strongly compatible with $\prec$. In the manifold case the Lorentzian distance (5) is already strongly compatible with $\prec$ even without the $\{y\}$ term, and the latter makes no difference, while for causal measure spaces that are neither Lorentzian manifolds nor causal sets we may still not have strong compatibility (for example, see the points p and q in Fig 1a).

Conversely, given a Lorentzian distance d on a set X one can ask to what extent it determines a unique partially ordered measure space structure on X. In general, many partial orders will be weakly compatible with d, but there is a unique strongly compatible one, given as seen above by $x \prec y$ iff $d(x, y) > 0$, which we will consider to be the order relation associated with d. A Lorentzian distance also almost defines a measure, and it does define it uniquely in two important cases. One is the case of a causal set. In a discrete partially ordered set the measure of any non-minimal element x can be determined as the Lorentzian distance $\mu(\{x\}) = d(y, x)$, where y is any element covered by x (i.e., such that $y \prec x$ but there is no other element between y and x). If x is a minimal element, its measure cannot be determined this way because it does not affect the value of any Lorentzian distance; however, if we can assume that μ is the counting measure, then $\mu(\{x\}) = 1$.

Another case in which a causal measure space structure is uniquely determined is that of a differentiable manifold with a Lorentzian distance. In any set X endowed with a Lorentzian distance, one can define the future and past of an element $x \in X$ using the unique strongly compatible partial order, or equivalently by directly setting $I^+(x) := \{y \in X \mid d(x, y) > 0\}$ and $I^-(x) := \{y \in X \mid x \in I^+(y)\}$, which in turn allows us to define future- and past-distinguishing sets with Lorentzian distances. Then, in the case of a distinguishing Lorentzian-distance manifold (M, d), it follows from the quoted results of Hawking *et al.* and Malament [1, 2] that, if two Lorentzian metrics g and g' on M are compatible with d they must be conformally related, $g' = \Omega^2 g$ for some positive function Ω on M. But if the volumes of all Alexandrov sets in the two metrics coincide, then in fact we must have $g' = g$.

Summarizing, what we have seen is that two distinguishing Lorentzian manifolds are isomorphic iff their corresponding causal measure spaces are isomorphic, and the latter holds if their Lorentzian distance spaces are isomorphic. This suggests that, to come up with a definition of closeness for all causal measure spaces that can separate all distinguishing Lorentzian manifolds, one can compare their Lorentzian distance functions.

3. The Gromov-Hausdorff closeness function for causal measure spaces

Using the Lorentzian distance we can define, similarly to what is done in Ref [14], a Gromov-Hausdorff function characterizing the closeness of any two causal measure spaces. We will say that two causal measure spaces $(X, \prec_X, \mathcal{A}_X, \mu_X)$ and $(Y, \prec_Y, \mathcal{A}_Y, \mu_Y)$ are ϵ-*close* iff there exist maps $\xi : X \to Y$ and $\zeta : Y \to X$ such that, for all x_1, $x_2 \in X$ and y_1, $y_2 \in Y$,

$$|d_X(x_1, x_2) - d_Y(\xi(x_1), \xi(x_2))| < \epsilon\,, \quad |d_Y(y_1, y_2) - d_X(\zeta(y_1), \zeta(y_2))| < \epsilon\,, \tag{6}$$

where d_X and d_Y are the Lorentzian distances defined as in (5) on X and Y, respectively. If some specific pair of maps (ξ, ζ) are such that these inequalities hold, we will say that X

and Y are ϵ-close with respect to (ξ, ζ). Notice that the two maps used to establish a "best match" between X and Y are not required to satisfy any condition, not even that of being continuous in any topology we might be using on X and Y, a priori. We then define the following Gromov-Hausdorff function for pairs of causal measure spaces:

$$d_{\rm GH}(X,Y) := \inf\{\epsilon \mid (X, \prec_X, \mathcal{A}_X, \mu_X) \text{ and } (Y, \prec_Y, \mathcal{A}_Y, \mu_Y) \text{ are } \epsilon\text{-close}\}\ . \tag{7}$$

It is trivial to see that this function is symmetric. It is also easy to show that it satisfies the triangle inequality, by the following argument. For any spaces X, Y and Z, if we call $d_{\rm GH}(X,Y) = \delta$ and $d_{\rm GH}(Y,Z) = \delta'$, there exist maps $\xi: X \to Y$ and $\zeta: Y \to X$ such that X and Y are δ-close with respect to (ξ, ζ), and maps $\xi': Y \to Z$ and $\zeta': Z \to Y$ such that Y and Z are δ'-close with respect to (ξ', ζ'). Consider then the maps defined by $\eta := \xi' \circ \xi : X \to Z$ and $\chi := \zeta \circ \zeta' : Z \to X$. For any x_1, $x_2 \in X$ and z_1, $z_2 \in Z$ we have

$$\begin{aligned}
&|d_X(x_1,x_2) - d_Z(\eta(x_1), \eta(x_2))| \\
&\equiv |d_X(x_1,x_2) - d_Y(\xi(x_1), \xi(x_2)) + d_Y(\xi(x_1), \xi(x_2)) - d_Z(\eta(x_1), \eta(x_2))| \\
&\leq |d_X(x_1,x_2) - d_Y(\xi(x_1), \xi(x_2))| + |d_Y(\xi(x_1), \xi(x_2)) - d_Z(\xi'(\xi(x_1)), \xi'(\xi(x_2)))| \\
&< \delta + \delta'\ ,
\end{aligned} \tag{8}$$

and similarly that $|d_Z(z_1,z_2) - d_X(\chi(z_1), \chi(z_2))| < \delta + \delta'$. In other words, for any X, Y and Z, $d_{\rm GH}(X,Z) \leq d_{\rm GH}(X,Y) + d_{\rm GH}(Y,Z)$. The function $d_{\rm GH}$ in (7) is therefore a pseudodistance on the space of all causal measure spaces.

However, in general $d_{\rm GH}$ may fail to be positive-definite and is therefore not a distance. In the next two sections we will examine the reasons why $d_{\rm GH}$ may vanish for non-isomorphic spaces, and show that the restriction of $d_{\rm GH}$ to a suitably defined class of causal measure spaces is a distance.

4. A non-local pseudodistance on a causal measure space

We begin by introducing two highly nonlocal pseudodistances on a causal measure space X, which will turn out to be useful tools for analyzing properties of this type of space. Given any elements x and $y \in X$, the functions

$$D^+(x,y) := \sup_{z\in X} \big|d(x,z) - d(y,z)\big|\ , \qquad D^-(x,y) := \sup_{z\in X} \big|d(z,x) - d(z,y)\big| \tag{9}$$

quantify the difference between the causal relations that x and y have with other elements of X in their futures or their pasts, respectively, as encoded in the Lorentzian distances between x and y and those other elements. We also define their combination

$$D(x,y) := \max\{D^+(x,y), D^-(x,y)\}\ , \tag{10}$$

which takes into account the Lorentzian distances between x and y and all other points in X. The triangle inequality for D^+ is easily verified:

$$\begin{aligned}
D^+(x,y) + D^+(y,z) &= \sup_{w\in X} \big|d(x,w) - d(y,w)\big| + \sup_{w\in X} \big|d(y,w) - d(z,w)\big| \\
&\geq \sup_{w\in X} \big[\big|d(x,w) - d(y,w)\big| + \big|d(y,w) - d(z,w)\big|\big] \\
&\geq \sup_{w\in X} \big|d(x,w) - d(z,w)\big| = D^+(x,z)\ ,
\end{aligned} \tag{11}$$

and similarly for $D^-(x,y)+D^-(y,z)$; therefore, for D we have

$$\begin{aligned} D(x,y)+D(y,z) &= \max\{D^+(x,y),D^-(x,y)\}+\max\{D^+(y,z),D^-(y,z)\} \\ &\geq \max\{D^+(x,y)+D^+(y,z),D^-(x,y)+D^-(y,z)\} \\ &\geq \max\{D^+(x,z),D^-(x,z)\} = D(x,z)\ , \end{aligned} \tag{12}$$

hence the triangle inequality holds for D as well.

If we don't impose any condition on the causal measure space, however, in general $D^\pm(x,y)$ may be infinite for most pairs of points. For example, in Minkowski space $D^\pm(x,y) = \infty$ for all $x\neq y$. In this case, the problem is avoided if we consider for example only the portion of Minkowski space between two parallel spacelike hyperplanes: in four dimensions, if the proper time separation (in the normal direction) is τ, then the volumes of all Alexandrov sets defined by pairs of points inside this region are bounded, $\mu(A(x,y)) \leq \frac{\pi}{24}\,\tau^4$, and therefore both $d(x,y)$ and $D(x,y)$ are also bounded by $\frac{\pi}{24}\,\tau^4$, for all x and y. More generally, to ensure that D has meaningful values we could require the existence of an upper bound on $\mu(A(x,y))$ for all pairs $x,\ y\in X$, but for later purposes from now on we will restrict ourselves to compact causal measure spaces; in particular, this will imply that the values of all $\mu(A(x,y))$, and therefore of all $D(x,y)$, are bounded above.

Notice that the function D is also not a distance, since in general it is not positive-definite. The main type of example of situation in which D cannot separate pairs of points in a causal measure space X is when X is non-distinguishing. If there exist two points $x\neq y\in X$ with $I^-(x) = I^-(y)$ and $I^+(x) = I^+(y)$, then $\forall z\in X$, $A(x,z) = A(y,z)$ and $A(z,x) = A(z,y)$; unless these two points have different measures $\mu(\{x\})\neq\mu(\{y\})$, which can happen in a discrete setting if μ is not simply the counting measure,[6] this implies that $D^\pm(x,y) = 0$ and D is not positive-definite. On the other hand, a sufficient condition for D to be positive-definite is that X be distinguishing and $(d,\prec)$ strongly compatible ($d(x,y)>0$ for all $x\prec y$). This observation will play a role in our discussion of the result in Sec 5, but here we will comment on another point related to D.

The structure available on a causal measure space $(X,\prec,\mathcal{A},\mu)$ allows us to define two natural topologies on X (in addition to the manifold topology available in a continuum-based Lorentzian geometry). The first one is the well-known Alexandrov topology $\mathcal{T}_{\rm A}$, for which a subbase is given by the futures and pasts $\{I^+(p),I^-(p)\mid p\in X\}$ (in other words, the open sets in $\mathcal{T}_{\rm A}$ are arbitrary unions of finite intersections of sets of this type, and they include all Alexandrov sets [18]); this topology only uses the partial order relation on X. The second one is the topology $\mathcal{T}_D$ induced by the pseudodistance D, for which a base is given by the balls $B_{x,r} = \{y\in X\mid D(x,y)<r\}$ (in other words, the open sets in $\mathcal{T}_D$ are arbitrary unions of sets of this type). One can then ask whether one of these topologies is stronger than the other one. The following examples show that in general they are incomparable:

(1) $\mathcal{T}_{\rm A}$ *can be stronger than* $\mathcal{T}_D$. The Alexandrov topology $\mathcal{T}_{\rm A}$ is stronger than the D-topology $\mathcal{T}_D$ if all open sets in $\mathcal{T}_D$ are open in $\mathcal{T}_{\rm A}$ but there is at least one past, future or Alexandrov set which is not an open set in $\mathcal{T}_D$. Consider the example of Fig 1a. Most of the open sets in each of the topologies are also open in the other topology, but the correspondence breaks

[6] If all pairs of points $x\neq y$ in X for which $I^\pm(x) = I^\pm(y)$ have $\mu(x)\neq\mu(y)$, we could say that X is distinguishing *as a causal measure space*, as opposed to just as a partially-ordered set.

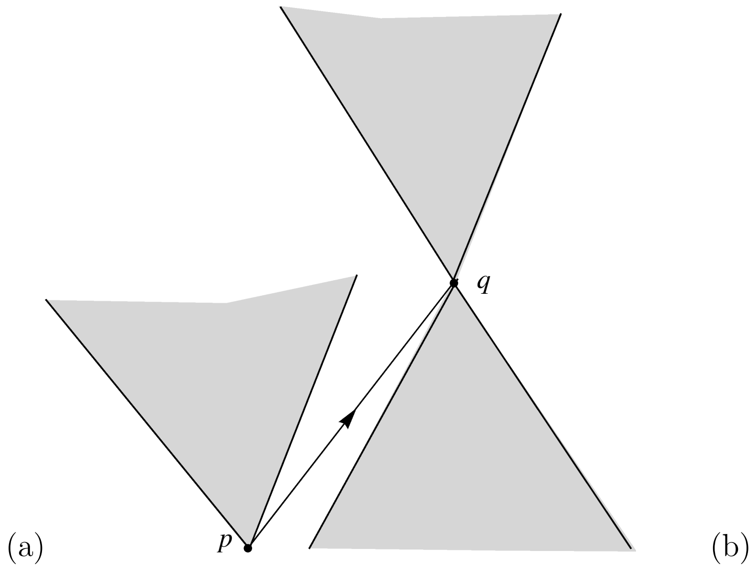

Figure 1: (a) An example of causal measure space in which the Alexandrov topology $\mathcal{T}_{\rm A}$ is stronger than the D topology $\mathcal{T}_D$. The set X consists of the shaded regions (which do not extend beyond the edge of the drawing), with the points p (the bottom vertex of the half lightcone) and q (the vertex of the double lightcone on the right) included. The causal relations and measure can be taken to be those induced on X as a subset of 2D Minkowski space with one added relation, $p \prec q$; this is not a Lorentzian manifold nor a discrete causal measure space. Notice also that $d(p,q) = \mu(\{q\}) = 0$, although $p \prec q$. (b) An example of discrete causal measure space in which $\mathcal{T}_D$ in stronger than $\mathcal{T}_{\rm A}$.

down around q. All open balls in $\mathcal{T}_D$ are open in $\mathcal{T}_{\rm A}$ (at least if we add $\{p\}$ as an open set in $\mathcal{T}_{\rm A}$, since for small enough $r > 0$, $B_{p,r} = \{p\}$), and all intervals $A(u,v)$ where both u and v are in $I^+(p)$ or in $I^-(q) \cup I^+(q)$ are also D-open. However, the other non-empty intervals in this poset are of the form $A(p,v) = \{q\} \cup A(q,v)$, with $v \in I^+(q)$; because of the presence of $\{q\}$, a set of this form is not in $\mathcal{T}_D$: any $\mathcal{T}_D$-open set containing q would also contain points in the lower half of the double lightcone.

(2) *$\mathcal{T}_D$ can be stronger than $\mathcal{T}_{\rm A}$.* The D-topology $\mathcal{T}_D$ is stronger than the Alexandrov topology $\mathcal{T}_{\rm A}$ if all open sets in $\mathcal{T}_{\rm A}$ are open in $\mathcal{T}_D$ but there is at least one D-ball which is not a union of Alexandrov sets, pasts and futures. Consider the example in Fig 1.b. This is a discrete, distinguishing causal measure space so, assuming that all elements have non-zero measure, each of them is open in $\mathcal{T}_D$. In particular, the subset $\{2\}$ is open in $\mathcal{T}_D$, but it is not in $\mathcal{T}_{\rm A}$ since every past and future containing $\{2\}$ contains $\{3\}$ as well.

5. The Gromov-Hausdorff function as a distance for causal measure spaces

We can now go back to the issue of the positive-definiteness of $d_{\rm GH}$. The examples in the previous section gave us some familiarity with the function $D(x,y)$. If D is positive-definite (or the topology $\mathcal{T}_D$ is Hausdorff) the Lorentzian distance d can be used to distinguish all elements of X, and we saw that D is positive-definite for all (distinguishing) spacetimes and all distinguishing causal sets. For our present purposes we are interested in properties of $d_{\rm GH}$ instead, but $d_{\rm GH}$ is defined using d, and quantifies the extent to which Lorentzian

distances in two causal measure spaces differ when we use appropriate maps to establish the best possible correspondence between them. These observations motivate the following proposition, our main result:

Proposition: The Gromov-Hausdorff pseudodistance $d_{\rm GH}$, when restricted to $\mathcal{T}_D$-compact causal measure spaces corresponding to distinguishing Lorentzian manifolds, is a distance.

Proof: Consider two causal measure spaces $(X, \prec_X, \mathcal{A}_X, \mu_X)$ and $(Y, \prec_Y, \mathcal{A}_Y, \mu_Y)$ corresponding to Lorentzian manifolds (M, g) and (N, h), as described in Section 2. We will show that if $d_{\rm GH}(X, Y) = 0$, then $(X, \prec_X, \mathcal{A}_X, \mu_X)$ and $(Y, \prec_Y, \mathcal{A}_Y, \mu_Y)$ are isomorphic, i.e., there exists a bijection $\xi: X \to Y$ such that both ξ and ξ^{-1} preserve the partial order relations and measures; the converse implication is obvious. Suppose then that $d_{\rm GH}(X, Y) = 0$. This means that $(X, \prec_X, \mathcal{A}_X, \mu_X)$ and $(Y, \prec_Y, \mathcal{A}_Y, \mu_Y)$ are ϵ-close for arbitrarily small ϵ. In particular, for any positive integer n there exist mappings

$$\xi_n: X \to Y\;, \quad \zeta_n: Y \to X \tag{13}$$

with respect to which $(X, \prec_X, \mathcal{A}_X, \mu_X)$ and $(Y, \prec_Y, \mathcal{A}_Y, \mu_Y)$ are $(1/n)$-close. We will use these sequences of mappings to construct an isomorphism between X and Y.

Start by picking countable dense subsets $\mathcal{C}_1$ in X and $\mathcal{D}_1$ in Y, in the D topology; since X and Y are D-compact metric spaces, such subsets always exist. For the same reason, given any $p \in \mathcal{C}_1$ and $q \in \mathcal{D}_1$, the sequences $\{\xi_n(p)\}$ and $\{\zeta_n(q)\}$ of images in Y and X, respectively, have (non-unique) convergent subsequences. Choose one such subsequence for each $p \in \mathcal{C}_1$ and for each $q \in \mathcal{D}_1$, $\{\xi_{n_k}(p)\}$ and $\{\zeta_{n_k}(q)\}$, and construct two sequences of maps

$$\xi_{(1,k)}: \mathcal{C}_1 \to Y\;, \quad \zeta_{(1,k)}: \mathcal{D}_1 \to X\;, \tag{14}$$

where, for each $p \in \mathcal{C}_1$ and $q \in \mathcal{D}_1$, $\{\xi_{(1,k)}(p)\}$ and $\{\zeta_{(1,k)}(q)\}$ are the chosen convergent subsequences of $\{\xi_n(p)\}$ and $\{\zeta_n(q)\}$, respectively. Thus, the sequences of maps in (14) converge on all of $\mathcal{C}_1$ and $\mathcal{D}_1$, and we call ξ and ζ the respective limit maps on $\mathcal{C}_1$ and $\mathcal{D}_1$,

$$\xi_{(1,k)}(p) \to \xi(p)\;, \quad \zeta_{(1,k)}(q) \to \zeta(q) \quad \text{as } k \to \infty\;. \tag{15}$$

Notice that by construction, for all p_1, $p_2 \in \mathcal{C}_1$ and all q_1, $q_2 \in \mathcal{D}_1$,

$$d_X(p_1, p_2) = d_Y(\xi(p_1), \xi(p_2))\;, \quad d_Y(q_1, q_2) = d_X(\zeta(q_1), \zeta(q_2))\;. \tag{16}$$

If we now add the two sets of limit points to $\mathcal{C}_1$ and $\mathcal{D}_1$, we obtain two new countable dense subsets $\mathcal{C}_2 \subset X$ and $\mathcal{D}_2 \subset Y$ and, using the same procedure as above, two new converging sequences of maps $\xi_{(2,k)}$ and $\zeta_{(2,k)}$ on $\mathcal{C}_2$ and $\mathcal{D}_2$; since we can choose $\xi_{(2,k)}(p) = \xi_{(1,k)}(p)$ when $p \in \mathcal{C}_1$, and $\zeta_{(2,k)}(q) = \zeta_{(1,k)}(q)$ when $q \in \mathcal{D}_1$, the limit maps are extensions of the ones in (15), and we will still call them ξ and ζ. This procedure can be iterated recursively to obtain, for each integer $i > 1$, two countable dense subsets of X and Y,

$$\mathcal{C}_i := \mathcal{C}_{i-1} \cup \{\lim_{k\to\infty} \zeta_{(i-1,k)}(q) \mid q \in \mathcal{D}_{i-1}\}\;, \quad \mathcal{D}_i := \mathcal{D}_{i-1} \cup \{\lim_{k\to\infty} \xi_{(i-1,k)}(p) \mid p \in \mathcal{C}_{i-1}\}\;, \tag{17}$$

and two corresponding sequences of maps

$$\xi_{(i,k)}: \mathcal{C}_i \to Y\;, \quad \zeta_{(i,k)}: \mathcal{D}_i \to X \tag{18}$$

such that, for each i and for each $p \in \mathcal{C}_i$, $\{\xi_{(i,k)}(p)\}$ is a suitably chosen subsequence of $\{\xi_n(p)\}$ and, for each $q \in \mathcal{D}_i$, $\{\zeta_{(i,k)}(q)\}$ is a suitably chosen subsequence of $\{\zeta_n(q)\}$, such that as $k \to \infty$ the maps $\xi_{(i,k)}$ and $\zeta_{(i,k)}$ converge to $\xi : \mathcal{C}_i \to Y$ and $\zeta : \mathcal{D}_i \to X$, respectively. By construction, for each i the limit maps preserve Lorentzian distances as in (16).

The next part of the construction is to extend the maps ξ and ζ to all of X and Y. To achieve this, first use a Cantor diagonal element argument applied to the double sequences of mappings $\xi_{(i,k)}$ and $\zeta_{(i,k)}$, and define the subsequences $\bar{\xi}_k := \xi_{(k,k)}$ and $\bar{\zeta}_k := \zeta_{(k,k)}$. These subsequences converge pointwise to mappings $\xi : \mathcal{C} \to \mathcal{D}$ and $\zeta : \mathcal{D} \to \mathcal{C}$, with

$$\mathcal{C} := \lim_{k\to\infty} \mathcal{C}_k = \bigcup_{k=1}^{\infty} \mathcal{C}_k \quad \text{and} \quad \mathcal{D} := \lim_{k\to\infty} \mathcal{D}_k = \bigcup_{k=1}^{\infty} \mathcal{D}_k \ . \tag{19}$$

In other words, we have found dense countable subsets $\mathcal{C} \subset X$ and $\mathcal{D} \subset Y$ and subsequences $\{\bar{\xi}_k\}$ of $\{\xi_n\}$ and $\{\bar{\zeta}_k\}$ of $\{\zeta_n\}$ such that, as $k \to \infty$,

$$\forall p \in \mathcal{C} \quad \bar{\xi}_k(p) \to \xi(p) \in \mathcal{D} \ , \quad \forall q \in \mathcal{D} \quad \bar{\zeta}_k(q) \to \zeta(q) \in \mathcal{C} \ , \tag{20}$$

with convergence understood in the D_Y and D_X topologies, respectively. By construction, the maps ξ and ζ preserve d_X and d_Y, i.e., the equalities in (16) hold in all of $\mathcal{C}$ and $\mathcal{D}$.

We now want to show that the image $\xi(\mathcal{C})$ is dense in Y, and $\zeta(\mathcal{D})$ is similarly dense in X. Since $\xi(\mathcal{C}) \subset \mathcal{D}$ and $\zeta(\mathcal{D}) \subset \mathcal{C}$, it is sufficient to prove that $\xi \circ \zeta(\mathcal{D})$ is dense in Y and $\zeta \circ \xi(\mathcal{C})$ is dense in X. Consider $\xi \circ \zeta(\mathcal{D})$, and suppose it is not dense in Y. Then there exists a point $q \in \mathcal{D}$ and some $\epsilon > 0$ such that $\xi \circ \zeta(\mathcal{D})$ is outside $B_{D_Y}(q, \epsilon)$. This implies that, for all $n \geq 1$, $(\xi \circ \zeta)^n(\mathcal{D})$ is also outside $B_{D_Y}(q, \epsilon)$, or $D_Y((\zeta \circ \xi)^n(r), q) > \epsilon$ for all $r \in \mathcal{D}$. In particular, $D_Y((\zeta \circ \xi)^n(q), q) > \epsilon$ and, since $(\xi \circ \zeta)$ preserves the distance D_Y, $D_Y((\xi \circ \zeta)^{n+m}(q), (\xi \circ \zeta)^m(q)) > \epsilon$ for all $m \geq 0$. However, because Y is compact, the sequence $\{(\xi \circ \zeta)^n(q)\}$ has a convergent subsequence, in contradiction with the statement that the distance between any two of its elements is greater than ϵ.

Finally, we extend the maps ξ and ζ from $\mathcal{C}$ and $\mathcal{D}$ to all of X and Y, respectively. Consider an element $r \in X \setminus \mathcal{C}$. Although we cannot simply invoke continuity and define $\xi(r)$ as the limit of the images of a sequence of elements of X that converge to r, because ξ has been defined as the pointwise limit of a sequence of maps which are not assumed to be continuous, we can see that in fact we can think of $\xi(r)$ as a limit; the argument is analogous to the one used to prove a similar statement in Ref [14], which we now summarize.

Consider a convex normal neighborhood $U \ni r$, and inside it pick an increasing sequence of elements of $\mathcal{C}$, $p_1 \prec p_2 \prec ... \prec r$, converging to r. Then the sequence of images $\{\xi(p_i)\}$ must have a unique accumulation point $s \in Y$, because it must have at least one such point since Y is compact, the fact that ξ preserves the Lorentzian distances implies that $\xi(p_i) \prec \xi(p_{i+1})$ in Y for all i, and an increasing sequence cannot have more than one accumulation point. Similarly, given a decreasing sequence of elements of $\mathcal{C}$, $q_1 \succ q_2 \succ ... \succ r$, also converging to r, the sequence of images $\{\xi(q_i)\}$ must have a unique accumulation point $s' \succeq s$. Now on the one hand we have that, for all i, $d(s, s') < d(\xi(p_i), \xi(q_i))$, and on the other hand $d(\xi(p_i), \xi(q_i)) = d(p_i, q_i) \to 0$, so $d(s, s') = 0$. This by itself does not allow us to to conclude that $s = s'$, because the two points could be null related but distinct. In that case, however, it would not be possible for the rate at which $d(p_i, q_i) \to 0$ to match the rate at which $d(\xi(p_i), \xi(q_i)) \to 0$, so we conclude that $s = s'$. We can now replace $\{p_i\}$ by any other

increasing sequence converging to r and $\{q_i\}$ by any other decreasing sequence also converging to r, and conclude that the common limit point does not depend on the choice of sequences; we therefore identify s with $\xi(r)$. Similarly, we can define $\zeta(s)$ for any $s \in Y \setminus \mathcal{D}$. Because these spaces arise from distinguishing Lorentzian manifolds, a d-isomorphism implies an isomorphism as causal measure spaces. •

6. Conclusions

In this paper we have introduced the concept of causal measure space, a structure that includes all Lorentzian manifolds without closed timelike curves and causal sets, as well as more general objects. We have then introduced a Lorentzian distance on any causal measure space and a Gromov-Hausdorff closeness function $d_{\rm GH}$ for pairs of causal measure spaces, based on their Lorentzian distances, and shown that $d_{\rm GH}$ is a true distance among spaces with Lorentzian distances, when restricted to the important case of spaces satisfying certain relatively simple conditions in terms of an auxiliary distance D on each space. One of the next steps along the lines of this work will be the application of this result to the causal-set approach to quantum gravity, so we will now comment briefly on the types of questions about causal measure spaces that will come up in this context.

In causal-set theory the fundamental objects in terms of which the dynamics is formulated are discrete, but one is interested in knowing which ones are manifoldlike in the sense that they are well approximated by Lorentzian manifolds at large scales and, for those, how uniquely each one of them determines the structure of its continuum counterpart. The operative definition of manifoldlike causal set is based on the notion of *faithful embedding*, a mapping $f: X \to M$ from a causal set $(X, \prec)$ to a Lorentzian manifold (M, g) such that: (i) The partial order relations are preserved, $f(x) \prec_M f(y)$ iff $x \prec_X y$; (ii) The embedded points $f(X)$ are distributed uniformly; and (iii) The mean spacing between embedded points is everywhere much smaller than any characteristic scales defined by the geometry of (M, g) [5]. This notion, to the extent that it is actually well defined, is a probabilistic one, and may be overly restrictive in the sense that it does not allow for departures from embeddability of the causal set, while it can be argued that allowing for small-scale obstructions would give the theory a much greater flexibility, with possibly significant consequences, without affecting the macroscopic Lorentzian-manifold interpretation. Our closeness function might allow a better definition of manifoldlike cusal set, based on the smallness of the value of $d_{\rm GH}(X, M)$, which would not be probabilistic and would allow for departures from strict embeddability.

One issue that will come up as part of this study is precisely that of characterizing the freedom left in the structure of (M, g) if it is close to a given causal set $(X, \prec)$ in the $d_{\rm GH}$ sense. Thus, the Gromov-Hausdorff function will be used in situations in which both causal measure spaces are either Lorentzian manifolds or large, manifoldlike causal sets, and we will be most interested in knowing that $d_{\rm GH}$ is positive-definite when restricted to Lorentzian manifolds as in our Proposition. We know that Lorentzian manifolds satisfy the conditions of that Proposition if they are distinguishing and $\mathcal{T}_D$-compact; it would then be useful to understand better the relationship between $\mathcal{T}_D$ and $\mathcal{T}_{\rm A}$ or the manifold topology, since there may be cases in which we can prove our result more directly without using the function D.

However, we expect much larger, manifoldlike causal sets that approximate causally well-behaved spacetimes to have relatively few pairs of elements with the same pasts and futures, and none in a suitable continuum limit [12]. At any rate, it is not as crucial for $d_{\rm GH}$ to be positive-definite when restricted to causal sets and, if needed, those large causal sets can always be made distinguishing by quotienting them out by the equivalence relation $(x \sim y) \Leftrightarrow (I^{\pm}(x) = I^{\pm}(y))$, without making them too trivial.

An interesting consequence of defining manifoldlike causal sets using $d_{\rm GH}$ is that it also gives a new meaning to another type of question about causal sets, namely what type of causal set is a good discrete counterpart for a given Lorentzian manifold. This question has a standard answer in connection with the faithful-embedding definition of manifoldlike causal sets, which is that the best way to obtain such a causal set is to sprinkle points uniformly at random in the Lorentzian manifold (using a Poisson point process), and use the partial order induced among them by the causal structure of the manifold. With a Gromov-Hausdorff-based definition of manifoldlike causal set we can ask which N-element causal sets X minimize the value of $d_{\rm GH}(X, M)$, with what probability a Poisson process will give a causal set that is close to the manifold in this sense and, looking ahead, whether the dynamics produces precisely those types of causal sets.

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